

# A positive solution to the Busemann-Petty problem in $\mathbb{R}^4$

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## Introduction

Motivated by basic questions in Minkowski geometry, H. Busemann and C. M. Petty posed ten problems about convex bodies in 1956 (see [BP]). The first problem, now known as the Busemann-Petty problem, states:

*If  $K$  and  $L$  are origin-symmetric convex bodies in  $\mathbb{R}^n$ , and for each hyperplane  $H$  through the origin the volumes of their central slices satisfy*

$$\text{vol}_{n-1}(K \cap H) < \text{vol}_{n-1}(L \cap H),$$

*does it follow that the volumes of the bodies themselves satisfy*

$$\text{vol}_n(K) < \text{vol}_n(L)?$$

The problem is trivially positive in  $\mathbb{R}^2$ . However, a surprising negative answer for  $n \geq 12$  was given by Larman and Rogers [LR] in 1975. Subsequently, a series of contributions were made to reduce the dimensions to  $n \geq 5$  by a number of authors (see [Ba], [Bo], [G2], [Gi], [Pa], and [Z1]). That is, the problem has a negative answer for  $n \geq 5$ . See [G3] for a detailed description. It was proved by Gardner [G1] that the problem has a positive answer for  $n = 3$ . The case of  $n = 4$  was considered in [Z1]. But the answer to this case in [Z1] is not correct. This paper presents the correct solution, namely, the Busemann-Petty problem has a positive solution in  $\mathbb{R}^4$ , which, together with results of other cases, brings the Busemann-Petty problem to a conclusion.

A key step to the solution of the Busemann-Petty problem is the discovery of the relation of the problem to intersection bodies by Lutwak [Lu]. An origin-symmetric convex body  $K$  in  $\mathbb{R}^n$  is called an intersection body if its radial function  $\rho_K$  is the spherical Radon transform of a nonnegative measure  $\mu$  on the unit sphere  $S^{n-1}$ . The value of the radial function of  $K$ ,  $\rho_K(u)$ , in the direction  $u \in S^{n-1}$ , is defined as the distance from the center of  $K$  to its boundary in that direction. When  $\mu$  is a positive continuous function,  $K$  is

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called the intersection body of a star body. The notion of intersection body was introduced by Lutwak [Lu] who proved that the Busemann-Petty problem has a positive answer if  $K$  is an intersection body in  $\mathbb{R}^n$ . Based on this relation, a positive answer to the Busemann-Petty problem in  $\mathbb{R}^3$  was given by Gardner [G1] who showed that all origin-symmetric convex bodies in  $\mathbb{R}^3$  are intersection bodies.

The relation of the Busemann-Petty problem to intersection bodies proved by Lutwak can be formulated as: A negative answer to the Busemann-Petty problem is equivalent to the existence of convex nonintersection bodies (see [G2] and [Z2]). The author attempted in [Z1] to give a negative answer for all dimensions  $\geq 4$  by trying to show that cubes in  $\mathbb{R}^n$  ( $n \geq 4$ ) are not intersection bodies (see Theorem 5.3 in [Z1]). However, there is an error in Lemma 5.1 of [Z1]. It affects only Theorems 5.3 and 5.4 there. The correct version of Theorem 5.3 is that no cube in  $\mathbb{R}^n$  ( $n > 4$ ) is an intersection body. This follows immediately from Theorem 6.1 of [Z1] which says that no generalized cylinder in  $\mathbb{R}^n$  ( $n > 4$ ) is an intersection body. Note that the proof of Theorem 6.1 in [Z1] holds for intersection bodies, although the definition of intersection body of a star body was the one used in [Z1]. Therefore, Theorem 5.4 in [Z1] should have stated: The Busemann-Petty problem has a negative solution in  $\mathbb{R}^n$  for  $n > 4$ .

In his important work [K1], Koldobsky applied the Fourier transform to the study of intersection bodies. In [K2], he showed that cubes in  $\mathbb{R}^4$  are intersection bodies. It was this result that exposed the error mentioned above and led to the present paper, which presents the correct solution to the Busemann-Petty problem in  $\mathbb{R}^4$ . One of the key ideas in the proof, previously employed by Gardner [G1], is the use of cylindrical coordinates in computing the inverse spherical Radon transform.

### 1. The inverse Radon transform on $S^3$ and intersection bodies in $\mathbb{R}^4$

The radial function  $\rho_L$  of a star body  $L$  is defined by

$$\rho_L(u) = \max\{r \geq 0 : ru \in L\}, \quad u \in S^{n-1}.$$

It is required in this paper that the radial function is continuous and even. For basic facts about star bodies and convex bodies, see [G3] and [S].

For a continuous function  $f$  on  $S^{n-1}$ , the spherical Radon transform  $Rf$  of  $f$  is defined by

$$(Rf)(u) = \int_{S^{n-1} \cap u^\perp} f(v) dv, \quad u \in S^{n-1},$$

where  $u^\perp$  is the  $(n-1)$ -dimensional subspace orthogonal to the unit vector  $u$ . Since the spherical Radon transform is self-adjoint, one can define the Radon transform  $R\mu$  for a measure  $\mu$  on  $S^{n-1}$  by

$$\langle R\mu, f \rangle = \langle \mu, Rf \rangle.$$

The intersection body  $IL$  of star body  $L$  is defined by

$$\rho_{IL}(u) = \text{vol}_{n-1}(L \cap u^\perp) = R\left(\frac{1}{n-1}\rho_L^{n-1}\right)(u), \quad u \in S^{n-1}.$$

An origin-symmetric convex body  $K$  is called the *intersection body* of a *star body* if there exists a star body  $L$  so that  $K = IL$ . That is, the inverse spherical Radon transform  $R^{-1}\rho_K$  is a positive continuous function. A slight extension of this definition is that an origin-symmetric convex body  $K$  is called an *intersection body* if the inverse spherical Radon transform  $R^{-1}\rho_K$  is a non-negative measure.

Let  $\Delta$  be the Laplacian on the unit sphere  $S^3$ . Helgason's inversion formula for the Radon transform  $R$  on  $S^3$  is (see [H, p. 161])

$$\frac{1}{16\pi^2}(1 - \Delta)RR = 1.$$

It implies that

$$(1) \quad R^{-1}\rho_K = \frac{1}{16\pi^2}R(1 - \Delta)\rho_K$$

for an origin-symmetric convex body  $K$  in  $\mathbb{R}^4$ . This formula shows that  $R^{-1}\rho_K$  is continuous when  $\rho_K$  is of class  $C^2$ . The following lemma provides an inversion formula which gives the positivity of  $R^{-1}\rho_K$ .

Let  $K$  be an origin-symmetric convex body in  $\mathbb{R}^4$ , and let  $A_u(z)$  be the volume of  $K \cap (zu + u^\perp)$ , where  $z$  is real and  $u \in S^3$ .

LEMMA 1. *If  $K$  is an origin-symmetric convex body in  $\mathbb{R}^4$  whose boundary is of class  $C^2$ , then*

$$(2) \quad (R^{-1}\rho_K)(u) = -\frac{1}{16\pi^2}A_u''(0), \quad u \in S^3.$$

*Proof.* By rotation, it suffices to prove (2) for the north pole of  $S^3$ . From Helgason's inversion formula (1), the inverse spherical Radon transform of  $\rho_K$ ,  $f = R^{-1}\rho_K$ , is a continuous function when  $\rho_K$  is of class  $C^2$ . Let

$$u = u(v, \phi) = (v \sin \phi, \cos \phi), \quad u \in S^3, \quad v \in S^2, \quad 0 \leq \phi \leq \pi,$$

and let  $\rho_K(v, \phi) = \rho_K(u)$  be the radial function of  $K$ . Define

$$\begin{aligned}\bar{\rho}_K(\phi) &= \int_{S^2} \rho_K(v, \phi) dv, \\ \bar{f}(\phi) &= \int_{S^2} f(u) dv, \\ r(v, \phi) &= \rho_K(v, \phi) \sin \phi, \\ \bar{r}(\phi) &= \bar{\rho}_K(\phi) \sin \phi.\end{aligned}$$

Consider  $\bar{\rho}_K$  and  $\bar{f}$  as functions on  $S^3$  which are  $\text{SO}(3)$  invariant. Since the spherical Radon transform is intertwining, we have  $\bar{\rho}_K = \text{R}\bar{f}$  (for a simple proof, see [G3, Th C.2.8]). From this and Lemma 2.1 in [Z1], or Theorem C.2.9 in [G3], we obtain

$$\bar{\rho}_K(\phi) = \frac{4\pi}{\sin \phi} \int_{\frac{\pi}{2}-\phi}^{\frac{\pi}{2}} \bar{f}(\psi) \sin \psi d\psi.$$

Taking the derivative on both sides of this equation gives

$$(\bar{\rho}_K(\phi) \sin \phi)' = 4\pi \bar{f}\left(\frac{\pi}{2} - \phi\right) \sin\left(\frac{\pi}{2} - \phi\right).$$

It follows that

$$4\pi \bar{f}(0) = \lim_{\phi \rightarrow \frac{\pi}{2}} \frac{(\bar{\rho}_K(\phi) \sin \phi)'}{\cos \phi} = -\bar{r}''\left(\frac{\pi}{2}\right).$$

Since  $\frac{1}{4\pi} \bar{f}(0)$  is the value of  $f$  at the north pole, we obtain

$$(3) \quad f(u_0) = -\frac{1}{16\pi^2} \bar{r}''\left(\frac{\pi}{2}\right),$$

where  $u_0$  is the north pole of  $S^3$ .

Consider the variable  $z$  defined by  $z = \rho_K \cos \phi$ . Then  $\tan \phi = \frac{r}{z}$ . Differentiating this equation and using  $\frac{1}{\cos^2 \phi} = 1 + \tan^2 \phi = 1 + \frac{r^2}{z^2}$  give

$$(4) \quad z^2 + r^2 = z \frac{dr}{d\phi} - r \frac{dz}{d\phi}.$$

This yields

$$(5) \quad \left. \frac{dz}{d\phi} \right|_{\phi=\frac{\pi}{2}} = -r(v, \frac{\pi}{2}).$$

Differentiating (4) gives

$$(6) \quad 2z \frac{dz}{d\phi} + 2r \frac{dr}{d\phi} = z \frac{d^2 r}{d\phi^2} - r \frac{d^2 z}{d\phi^2}.$$

From (5),

$$(7) \quad \left. \frac{dr}{d\phi} \right|_{\phi=\frac{\pi}{2}} = \left. \frac{dr}{dz} \frac{dz}{d\phi} \right|_{\phi=\frac{\pi}{2}} = -r \left. \frac{dr}{dz} \right|_{z=0}.$$

From (6) and (7),

$$(8) \quad \left. \frac{d^2 z}{d\phi^2} \right|_{\phi=\frac{\pi}{2}} = 2r \left. \frac{dr}{dz} \right|_{z=0}.$$

From (5), (8), and

$$\frac{d^2 r}{d\phi^2} = \frac{d^2 r}{dz^2} \left( \frac{dz}{d\phi} \right)^2 + \frac{dr}{dz} \frac{d^2 z}{d\phi^2},$$

we have

$$(9) \quad \begin{aligned} \left. \frac{d^2 r}{d\phi^2} \right|_{\phi=\frac{\pi}{2}} &= \left. \frac{d^2 r}{dz^2} \right|_{z=0} r(v, \frac{\pi}{2})^2 + 2r(v, \frac{\pi}{2}) \left( \left. \frac{dr}{dz} \right|_{z=0} \right)^2 \\ &= \left( r^2 \frac{d^2 r}{dz^2} \right)_{z=0} + \left( 2r \left( \frac{dr}{dz} \right)^2 \right)_{z=0} \\ &= \frac{1}{3} \left. \frac{d^2 r^3}{dz^2} \right|_{z=0}. \end{aligned}$$

Integrating both sides of (9) over  $S^2$  with respect to  $v$  gives

$$\int_{S^2} \left. \frac{d^2 r}{d\phi^2} (v, \phi) \right|_{\phi=\frac{\pi}{2}} dv = \frac{1}{3} \int_{S^2} \left. \frac{d^2 r^3}{dz^2} (v, z) \right|_{z=0} dv.$$

Since  $K$  has  $C^2$  boundary, one can interchange the second order derivative and the integral on each side of the last equation. We obtain

$$\left. \frac{d^2}{d\phi^2} \bar{r}(\phi) \right|_{\phi=\frac{\pi}{2}} = \frac{d^2}{dz^2} \left( \frac{1}{3} \int_{S^2} r^3(v, z) dv \right)_{z=0}.$$

Note that the 3-dimensional volume of the intersection of the hyperplane  $x_4 = z$  with the convex body  $K$ , denoted by  $A_{u_0}(z)$ , is given by

$$A_{u_0}(z) = \frac{1}{3} \int_{S^2} r^3(v, z) dv.$$

Therefore, we have

$$(10) \quad \bar{r}''\left(\frac{\pi}{2}\right) = A''_{u_0}(0).$$

Formula (2) follows from (3) and (10). □

Recently, Gardner, Koldobsky and Schlumprecht [GKS] have generalized the formula (2) to  $n$  dimensions by using techniques of the Fourier transform. A different proof of their formulas is given by Barthe, Fradelizi and Maurey [BFM].

**THEOREM 2.** *If  $K$  is an origin-symmetric convex body in  $\mathbb{R}^4$  whose boundary is of class  $C^2$  and has positive curvature, then  $K$  is an intersection body of a star body.*

*Proof.* By the Brunn-Minkowski inequality and the strict convexity of  $K$ ,  $A(z)^{\frac{1}{3}}$  is strictly concave. When one slices a symmetric convex body by parallel hyperplanes, the central section has maximal volume. Hence,  $A'(0) = 0$ . It follows that

$$A''(0) = 3A(0)^{\frac{2}{3}}(A(z)^{\frac{1}{3}})''_{z=0} < 0.$$

By Lemma 1,  $R^{-1}\rho_K$  is a positive continuous function. Therefore,  $K$  is the intersection body of a star body.  $\square$

When a convex body is identified with its radial function, the class of intersection bodies is closed under the uniform topology. Since every origin-symmetric convex body can be approximated by origin-symmetric convex bodies whose boundaries are of class  $C^2$  and have positive curvatures, we obtain:

**THEOREM 3.** *All origin-symmetric convex bodies in  $\mathbb{R}^4$  are intersection bodies.*

Theorem 3 is proved for convex bodies of revolution by Gardner [G2] and by Zhang [Z1], and is proved for cubes and other special cases by Koldobsky [K2]. In higher dimensions, the situation is different. For example, it is proved by Zhang [Z1] that generalized cylinders in  $\mathbb{R}^n$  ( $n > 4$ ) are not intersection bodies, and is proved by Koldobsky [K1] that the unit balls of finite dimensional subspaces of an  $L_p$  space,  $1 \leq p \leq 2$ , are intersection bodies. In three dimensions, Gardner [G1] proved that all origin-symmetric convex bodies in  $\mathbb{R}^3$  are intersection bodies. One can also prove this by Theorem 3 and a result of Fallert, Goodey and Weil [FGW] which says that central sections of intersection bodies are again intersection bodies. An intersection body may not be the intersection body of a star body. It is shown by Zhang [Z4] that no polytope in  $\mathbb{R}^n$  ( $n > 3$ ) is an intersection body of a star body. Campi [C] is able to prove a complete result which says that no polytope in  $\mathbb{R}^n$  ( $n > 2$ ) is an intersection body of a star body.

## 2. A positive solution to the Busemann-Petty problem in $\mathbb{R}^4$

The following relation of the Busemann-Petty problem to intersection bodies was proved by Lutwak [Lu].

THEOREM 4 (Lutwak). *The Busemann-Petty problem has a positive solution if the convex body with smaller cross sections is an intersection body.*

From Theorems 3 and 4, we conclude:

THEOREM 5. *The Busemann-Petty problem in  $\mathbb{R}^4$  has a positive solution.*

From Theorem 3 and Corollary 2.19 in [Z2], we have the following corollary about the maximal cross section of a convex body.

COROLLARY 6. *If  $K$  is an origin-symmetric convex body in  $\mathbb{R}^4$ , then*

$$(11) \quad \text{vol}_4(K)^{\frac{3}{4}} \leq \frac{3}{8}(\sqrt{2}\pi)^{\frac{1}{2}} \max_{u \in S^3} \text{vol}_3(K \cap u^\perp)$$

*with equality if and only if  $K$  is a ball.*

Inequality (11) implies that, in  $\mathbb{R}^4$ , balls attain the minmax of the volume of central hyperplane sections of origin-symmetric convex bodies with fixed volume. The corresponding inequality in  $\mathbb{R}^3$  to inequality (11) was proved by Gardner (see [G3, Th. 9.4.11]). However, it is no longer the case in higher dimensions at least for  $n \geq 7$ . Ball [Ba] showed that cubes are counterexamples for  $n \geq 10$ . Giannopoulos [Gi] showed that certain cylinders are counterexamples for  $n \geq 7$ . The following question, known as the slicing problem, has been of interest (see [MP] for details):

*Does there exist a positive constant  $c$  independent of the dimension  $n$  so that*

$$\text{vol}_n(K)^{\frac{n-1}{n}} \leq c \max_{u \in S^{n-1}} \text{vol}_{n-1}(K \cap u^\perp)$$

*for every origin-symmetric convex body  $K$  in  $\mathbb{R}^n$ ?*

## 3. The generalized Busemann-Petty problem

Besides considering hyperplane sections, one can also consider intermediate sections of convex bodies. For a fixed integer  $1 < i < n$ , the Busemann-Petty problem has the following generalization (see Problem 8.2 in [G3]):

*If  $K$  and  $L$  are origin-symmetric convex bodies in  $\mathbb{R}^n$ , and for every  $i$ -dimensional subspace  $H$  the volumes of sections satisfy*

$$\text{vol}_i(K \cap H) < \text{vol}_i(L \cap H),$$

does it follow that the volumes of the bodies themselves satisfy

$$\text{vol}_n(K) < \text{vol}_n(L)?$$

When  $i = n - 1$ , this is the Busemann-Petty problem. It turns out that the solution to the generalized Busemann-Petty problem depends strongly on the dimension  $i$  of the sections of convex bodies. It is proved by Bourgain and Zhang [BoZ] that the solution is negative when  $3 < i < n$ . The generalized Busemann-Petty problem has a positive solution when  $K$  belongs to a certain class of convex bodies, called  $i$ -intersection bodies, which contains all intersection bodies (see Theorem 5 in [Z3] and Lemma 6.1 in [GrZ]). In particular, when  $K$  is an intersection body, the generalized Busemann-Petty problem has a positive solution. From this fact and Theorem 3, we have:

**THEOREM 7.** *The generalized Busemann-Petty problem in  $\mathbb{R}^4$  has a positive solution.*

It might be still true that the generalized Busemann-Petty problem has a positive solution when  $i = 2, 3$ , and  $n \geq 5$ . This remains open.

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